Mathematical Induction

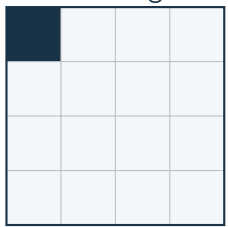
COMP2711 Discrete Mathematical Tools

Mingxun Zhou (mingxunz@ust.hk)

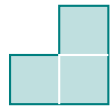
Case study 1: Can you tile this board?

For any integer $k \geq 1$, one square is missing from a $2^k \times 2^k$ board.

Can you cover the remaining board with L-shaped tiles?



Example: 4×4



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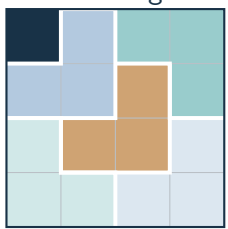

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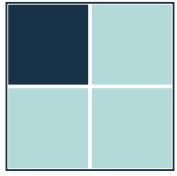

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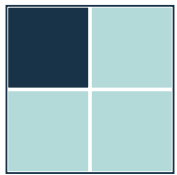
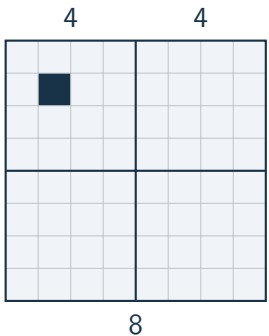


One tile is enough.

Case study 1: Four smaller boards



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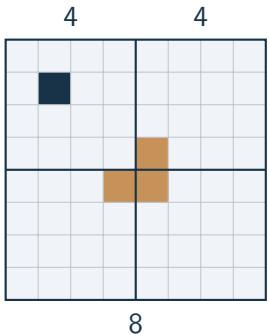

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Add one tile at the center.

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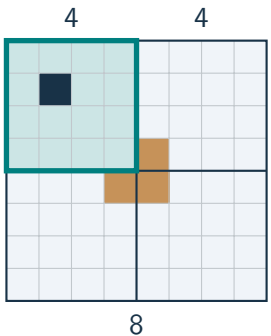

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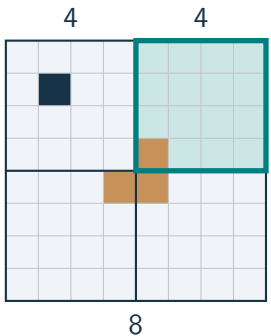

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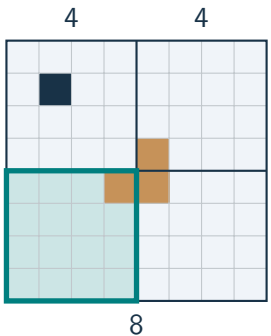

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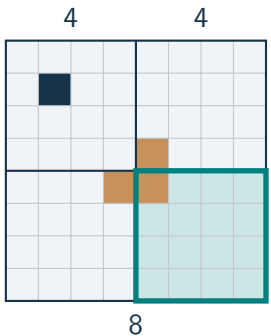

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Case study 1: The proof for general cases

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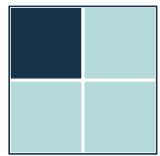

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Case study 1: The proof for general cases

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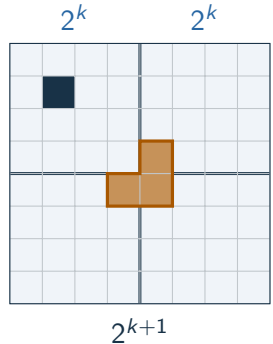
A 2×2 board with one hole needs one tile.

Double the size

If we can tile every $2^k \times 2^k$ board:

four smaller solutions + one central tile

cover a $2^{k+1} \times 2^{k+1}$ board.



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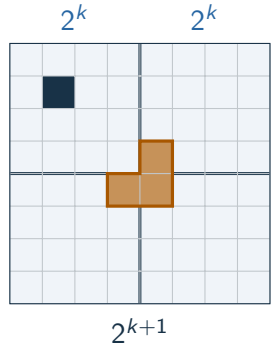

Therefore, for every $k \geq 1$, a $2^k \times 2^k$ board with any one square missing can be tiled.

Lecture roadmap

1. **Basic induction**
   Prove the first case and the step to the next.

2. **Strong induction**
   Use any earlier cases to prove the next.

3. **Well-ordering principle**
   Rule out a smallest counterexample.

Contents

1. **Basic induction**

2. Strong induction

3. Well-ordering principle

Mathematical induction

MUST UNDERSTAND

Induction principle

To prove $P(n)$ for every integer $n \geq b$:

1. **Base case:** prove $P(b)$.
2. **Inductive step:** for every $k \geq b$, prove $P(k) \Rightarrow P(k+1)$.

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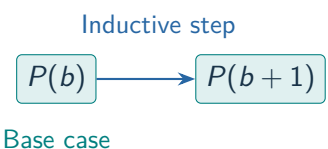

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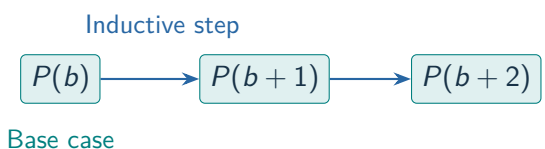

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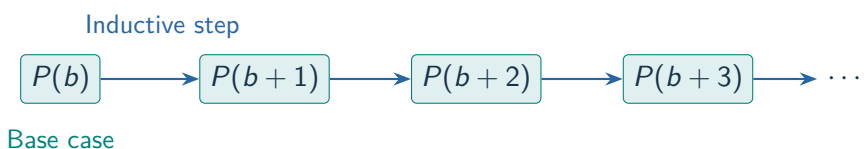

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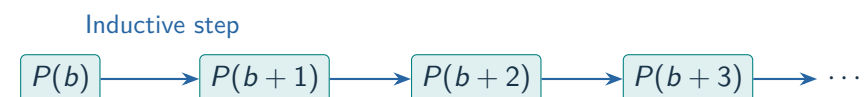

Case study 2: Odd-number sum

To prove

For every integer $n \geq 1$,

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Fix $k \geq 1$ and assume

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Thus the formula holds for every integer $n \geq 1$, by induction.

Case study 3: Choose your base case appropriately

For which starting value can we prove $2^n < n!$?

n	1	2	3	4
2^n	2	4	8	16
$n!$	1	2	6	24

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For every integer $n \geq 4$, $2^n < n!$.

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Claim

For every integer $n \geq 4$, $2^n < n!$.

Base: $16 < 24$. Assume $2^k < k!$ for $k \geq 4$.

$$2^{k+1} = 2 \times 2^k < 2k! \ < (k+1)k! = (k+1)!.$$

Case study 4: Divisibility (practice)

To prove

For every integer $n \geq 1$, $3 \mid (n^3 - n)$.

Base: $1^3 - 1 = 0$. **Another example:** $2^3 - 2 = 6$.

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Therefore, 3 divides their sum.

Contents

1. Basic induction

2. **Strong induction**

3. Well-ordering principle

Basic induction and strong induction

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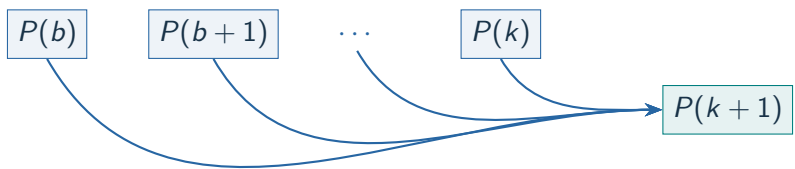

Strong induction

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Strong induction principle

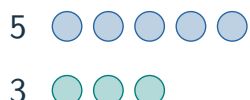
Prove $P(b)$. For every $k \geq b$, prove

$$\underbrace{P(b), P(b+1), \ldots, P(k)}_{\text{assume all earlier cases}} \implies P(k+1).$$

Then $P(n)$ holds for every integer $n \geq b$.

Case study 5: Two-pile Nim

- Two players take turns removing stones from **one pile**.
- Remove any positive number, up to the whole pile.
- Whoever takes the last stone wins.



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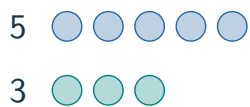
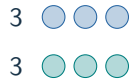


You move first. How would you play?

Case study 5: An example game

Start with $(3, 3)$. Your opponent moves first.



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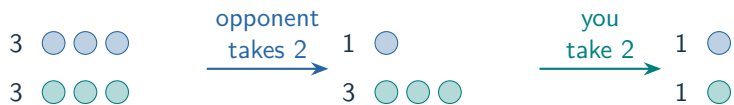

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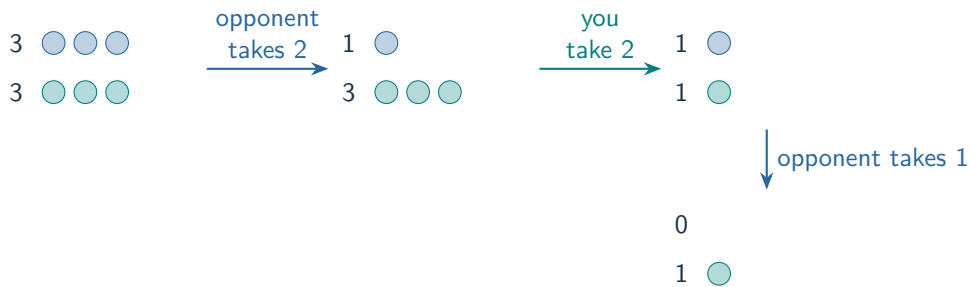

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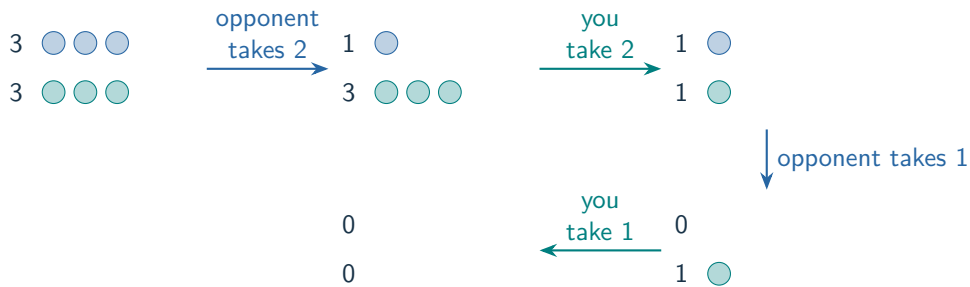
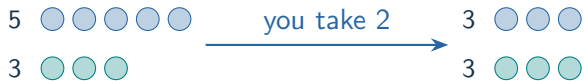


Case study 5: Match your opponent's move

Make the piles equal. Then copy each move in the other pile.



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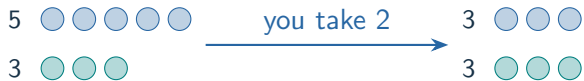

Your opponent's next move

Leaves stones in the pile: copy the removal. Smaller equal piles.

Empties a pile: take the remaining pile. You win.

Case study 5: Why matching wins

Claim

(n, n) is losing for the first player, for every $n \geq 1$.

Base: $(1, 1)$. The first player takes one stone; the second takes the last.

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Now consider $(n + 1, n + 1)$**.**

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Case 1: The first player empties a pile.
The second empties the other pile and wins.

Case 2: The first player takes $1 \leq m \leq n$ stones.
The second matches the move, leaving

$$(n + 1 - m, n + 1 - m), \qquad 1 \leq n + 1 - m \leq n.$$

It is the first player's turn again. By the hypothesis, the first player loses.

Case study 6: Sums of Fibonacci numbers

Fibonacci sequence

$F_1 = 1$, $F_2 = 2$, and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 2$.

1 2 3 5 8 13 21 34 55 89

Claim

Every positive integer can be expressed as a sum of distinct Fibonacci numbers.

Case study 6: Proof – find a smaller case

Base: $1 = F_1$ and $2 = F_2$.

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Strong-induction hypothesis

Fix $n \geq 3$. Every positive integer smaller than n is a sum of distinct Fibonacci numbers.

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Strong-induction hypothesis

Fix $n \geq 3$. Every positive integer smaller than n
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Choose consecutive Fibonacci numbers that bracket n: $F_k \leq n < F_{k+1}$.

If $n = F_k$, we are done.
Otherwise, consider the remainder $r = n - F_k > 0$.

Case study 6: Proof – find a smaller case

Base: $1 = F_1$ and $2 = F_2$.

Strong-induction hypothesis

Fix $n \geq 3$. Every positive integer smaller than n is a sum of distinct Fibonacci numbers.

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Subtract F_k from both sides:

$$\underbrace{n < F_{k+1}}_{\text{choice of } F_k} \implies r = n - F_k < \underbrace{F_{k+1} - F_k}_{\substack{=F_{k-1} \\ \text{Fibonacci recurrence}}} .$$

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Each term is at most $r < F_{k-1}$, so none is F_k.

Add F_k

$$n = F_k + \underbrace{F_{i_1} + \cdots + F_{i_t}}_{r} \text{ is a sum of distinct Fibonacci numbers.}$$

Contents

1. Basic induction

2. Strong induction

3. **Well-ordering principle**

Well-ordering: Proving a set is empty

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Contradiction: S cannot have a smallest element.
Therefore, S must be empty.

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Idea

Construct another solution (x', y', z') from (x, y, z) with $z' < z$.
This contradicts our choice of a solution with smallest z.

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Therefore, the solution set must be empty.

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Dividing both sides by 9 gives

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 3\left(\frac{z}{3}\right)^2.$$

We just need to argue that $x/3$, $y/3$, and $z/3$ are all integers.

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We have $x^2 + y^2 = 3z^2$ for positive integers x, y, z.

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Taking remainders modulo 3 on both sides,

$$x^2 + y^2 \equiv 3z^2 \equiv 0 \pmod 3.$$

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$x^2 + y^2 \pmod 3$

$x \backslash y$	0	1	2
0	**0**	1	1
1	1	2	2
2	1	2	2

Check all pairs of remainders in $\{0, 1, 2\}$.

Only $x \equiv y \equiv 0 \pmod 3$ works.

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Step 2: $3 \mid z$.

By Step 1, we further know that $9 \mid x^2$ and $9 \mid y^2$.

So

$$3 \mid \frac{x^2 + y^2}{3} = z^2.$$

Thus $3 \mid z$.

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Thus $x/3$, $y/3$, and $z/3$ are positive integers satisfying

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Contradiction: no smallest solution exists.
By the well-ordering principle, no positive integer solution exists.

Summary

1. **Basic induction**
   A base case and the implication $P(k) \Rightarrow P(k+1)$.

2. **Strong induction**
   Assume all earlier cases to prove the next.

3. **Well-ordering principle**
   Choose a smallest counterexample; use smaller cases to contradict it.

Review exercises

1. Prove the geometric-sum identity for $n \geq 0$, $r \neq 1$:

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

2. A proof checks $P(4)$ and proves $P(k) \Rightarrow P(k+2)$. What is missing if the target is every integer $n \geq 4$?

Solutions are in the course note.