Graph 1

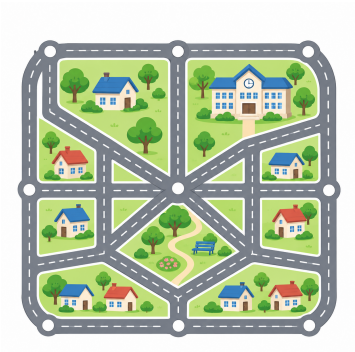
COMP2711 Discrete Mathematical Tools

Mingxun Zhou (mingxunz@ust.hk)

Graphs are everywhere



Social networks



Road networks

SECTION 1

Graph models

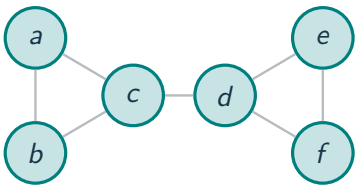
Vertices and edges

MUST UNDERSTAND

Simple undirected graph

$G = (V, E)$: vertices V and edges E.

Usually: $|V| = n$ vertices, $|E| = m$ edges.



$$\boldsymbol{V = \{a, b, c, d, e, f\}}$$

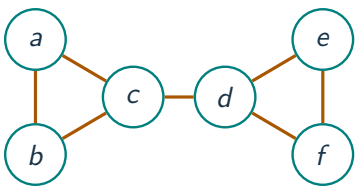
Vertices and edges

MUST UNDERSTAND

Simple undirected graph

$G = (V, E)$: vertices V and edges E.

Usually: $|V| = n$ vertices, $|E| = m$ edges.



$V = \{a, b, c, d, e, f\}$

$\boldsymbol{E = \{ab, ac, bc, cd, de, df, ef\}}$

Vertices and edges

MUST UNDERSTAND

Simple undirected graph

$G = (V, E)$: vertices V and edges E.

Usually: $|V| = n$ vertices, $|E| = m$ edges.

Excluded in a simple graph



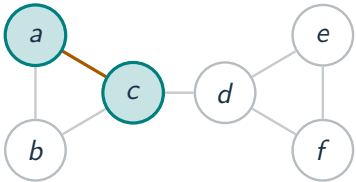
Loop



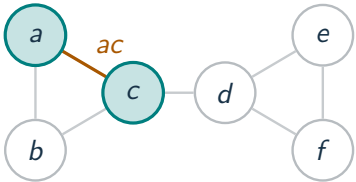
Multiple edges

Adjacency and incidence

Adjacent: ***a*** and ***c*** are joined by an edge.



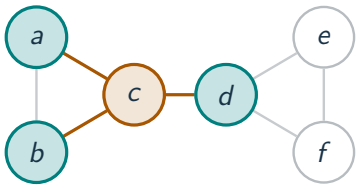
Adjacency and incidence



Adjacent: ***a*** and ***c*** are joined by an edge.

Incident: edge *ac* meets vertices ***a*** and ***c***.

Adjacency and incidence



Adjacent: ***a*** and ***c*** are joined by an edge.

Incident: edge *ac* meets vertices ***a*** and ***c***.

Neighbors of *c*:

$$N(c) = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}\}.$$

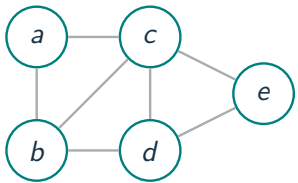
Subgraphs

Subgraph

A graph formed by keeping some vertices and some edges of G.

$$H = (V', E'), \qquad V' \subseteq V, \quad E' \subseteq E.$$

Every edge in E' must have both endpoints in V'.



G

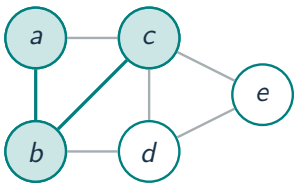
Subgraphs

Subgraph

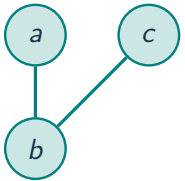
A graph formed by keeping some vertices and some edges of G.

$$H = (V', E'), \qquad V' \subseteq V, \quad E' \subseteq E.$$

Every edge in E' must have both endpoints in V'.



G



H: keep a, b, c and edges ab, bc.

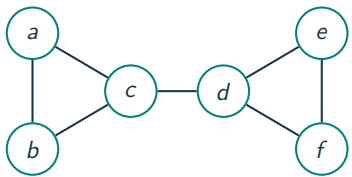
Degree

MUST UNDERSTAND

Degree

In a simple graph, the degree of v is its number of neighbors:

$$\deg(v) = |N(v)|.$$



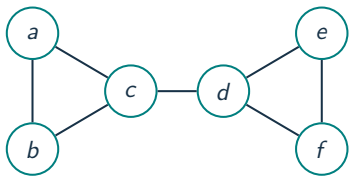
Degree

MUST UNDERSTAND

Degree

In a simple graph, the degree of v is its number of neighbors:

$$\deg(v) = |N(v)|.$$



Vertex	a	b	c	d	e	f
Degree	2	2	3	3	2	2

Directed vs. undirected

Undirected

Mutual friendship: $\{a, b\}$.



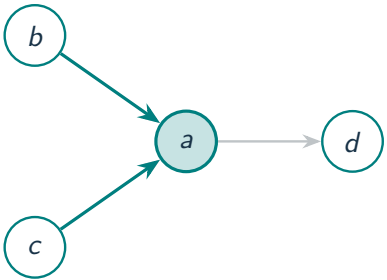
Directed

An edge (a, b) goes from a to b.



I follow Elon Musk on Twitter.
He didn't follow me back.

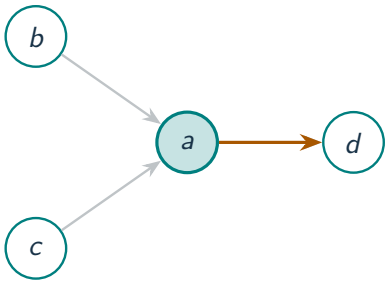
Directed degree



In-degree: number of incoming edges.

$\deg^-(a) = 2.$

Directed degree



In-degree: number of incoming edges.

$$\deg^-(a) = 2.$$

Out-degree: number of outgoing edges.

$$\deg^+(a) = 1.$$

Important Fact 1: Handshaking theorem

MUST UNDERSTAND

Theorem

In a finite undirected graph, **the sum of all vertex degrees is twice the number of edges.**

Important Fact 1: Handshaking theorem

MUST UNDERSTAND

Theorem

In a finite undirected graph, **the sum of all vertex degrees is twice the number of edges.**

$$\sum_{v \in V} \deg(v) = 2m.$$

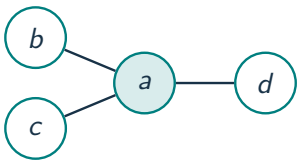
Important Fact 1: Handshaking theorem

MUST UNDERSTAND

Theorem

In a finite undirected graph, **the sum of all vertex degrees is twice the number of edges.**

$$\sum_{v \in V} \deg(v) = 2m.$$



$m = 3$.

$$3 + 1 + 1 + 1 = 6 = 2 \times 3.$$

Proof. Every edge contributes 1 to the degree of each endpoint.

Important Fact 2: Directed degree sums

Directed graph

The total in-degree and total out-degree both equal the number of edges.

Important Fact 2: Directed degree sums

Directed graph

The total in-degree and total out-degree both equal the number of edges.

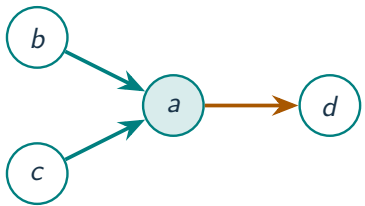
$$\sum_{v \in V} \deg^-(v) = m = \sum_{v \in V} \deg^+(v).$$

Important Fact 2: Directed degree sums

Directed graph

The total in-degree and total out-degree both equal the number of edges.

$$\sum_{v \in V} \deg^-(v) = m = \sum_{v \in V} \deg^+(v).$$



$m = 3$.

In: $2 + 0 + 0 + 1 = 3$,

Out: $1 + 1 + 1 + 0 = 3$.

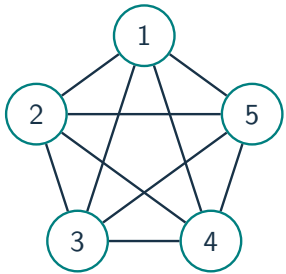
Proof. Each edge contributes 1 outgoing and 1 incoming.

SECTION 2

Important types of graphs

Complete graphs

Everyone is friends with everyone else.



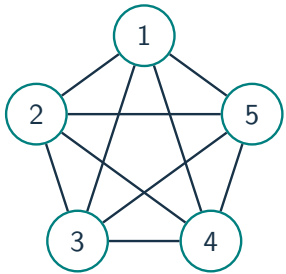
K_5

Complete graph K_n

Every pair of distinct vertices is adjacent.

Complete graphs

Everyone is friends with everyone else.



K_5

Complete graph K_n

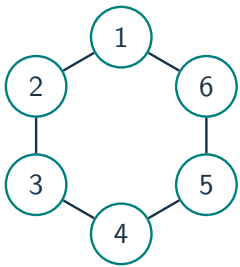
Every pair of distinct vertices is adjacent.

$$\deg(v) = n - 1, \qquad m = \binom{n}{2}.$$

For K_5: degree 4, edges 10.

Cycle graphs

A ring network: each device links to its two neighbors.



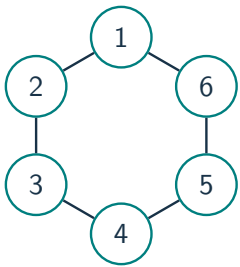
C_6

Cycle graph C_n, $n \geq 3$

Edges join consecutive vertices around a ring, including the last to the first.

Cycle graphs

A ring network: each device links to its two neighbors.



C_6

Cycle graph C_n, $n \geq 3$

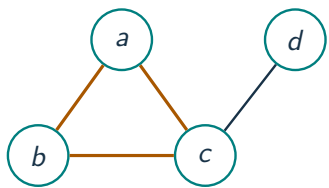
Edges join consecutive vertices around a ring, including the last to the first.

$$\deg(v) = 2, \qquad m = n.$$

Cycles in graphs

Undirected cycle

Follow at least three edges back to the start, without repeating other vertices.

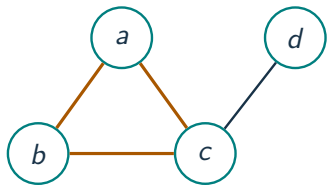


a, b, c, a

Cycles in graphs

Undirected cycle

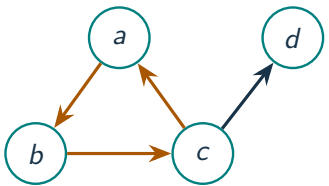
Follow at least three edges back to the start, without repeating other vertices.



a, b, c, a

Directed cycle

Follow the **arrows** back to the start, without repeating other vertices.



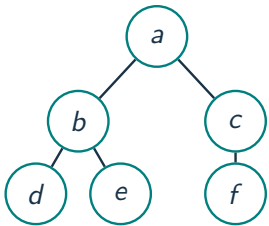
$a \to b \to c \to a$

Acyclic graphs: Trees and DAGs

Acyclic graphs are extremely important and common.
Acyclic means no cycles.

Tree (undirected)

A **connected** undirected graph with no cycles.

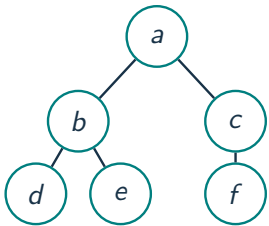


Acyclic graphs: Trees and DAGs

Acyclic graphs are extremely important and common.
Acyclic means no cycles.

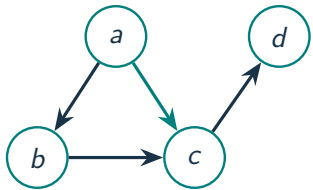
Tree (undirected)

A **connected** undirected graph with no cycles.



Directed acyclic graph (**DAG**)

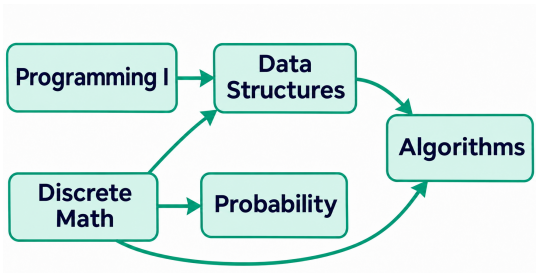
A directed graph with no directed cycles.



DAGs: Course prerequisites

DAGs are commonly used to represent dependencies.

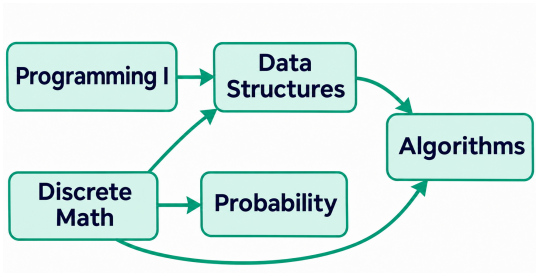
$A \to B$: complete course A before taking course B.



DAGs: Course prerequisites

DAGs are commonly used to represent dependencies.

$A \to B$: complete course A before taking course B.



Think about it: What if the prerequisite graph contains a **directed cycle**?

Think about it: How many edges?

What is the **maximum number of edges** in an **acyclic** simple graph with $n \geq 1$ vertices?

Consider the undirected and directed cases separately.

Think about it: How many edges?

What is the **maximum number of edges** in an **acyclic** simple graph with $n \geq 1$ vertices?

Consider the undirected and directed cases separately.

Undirected: $\boldsymbol{n-1}$ Directed: $\boldsymbol{\frac{n(n-1)}{2}}$.

Why?

Bipartite graphs

Definition

The vertices are split into **two parts**.
Edges go **only between** the two parts.

$$V = L \cup R, \qquad L \cap R = \varnothing,$$

$$E \subseteq \{\{x, y\} : x \in L,\ y \in R\}.$$

Bipartite graphs

Definition

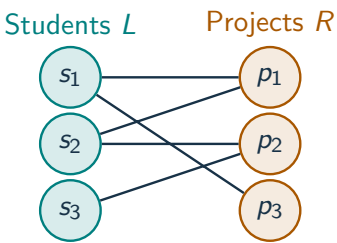
The vertices are split into **two parts**.
Edges go **only between** the two parts.

$$V = L \cup R, \qquad L \cap R = \varnothing,$$

$$E \subseteq \{\{x, y\} : x \in L,\ y \in R\}.$$

Example: Students and projects

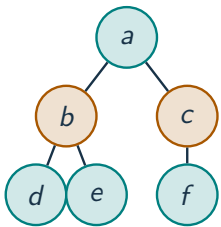
An edge means a student is interested in a project.



Bipartite graphs: Trees and cycles

Trees

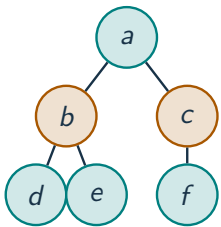
Every tree is **bipartite**.



Bipartite graphs: Trees and cycles

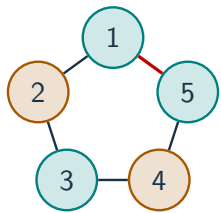
Trees

Every tree is **bipartite**.



Odd cycles

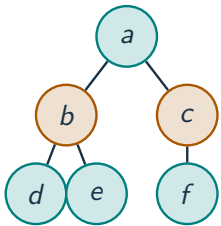
Not bipartite.

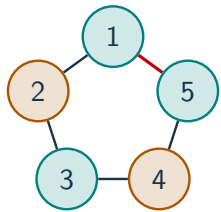


C_5

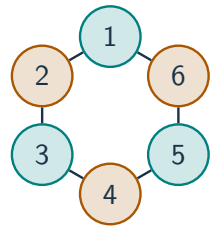
Bipartite graphs: Trees and cycles

Trees	Odd cycles	Even cycles
Every tree is **bipartite**.	**Not bipartite.**	**Bipartite.**





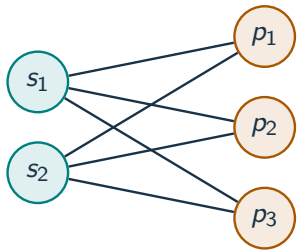
C_5



C_6

Complete bipartite graphs

Every student is interested in every project.



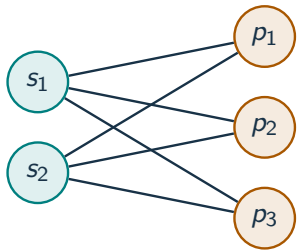
$K_{2,3}$

Complete bipartite graph $K_{r,s}$

All cross-edges; no edges within a part.

Complete bipartite graphs

Every student is interested in every project.



$K_{2,3}$

Complete bipartite graph $K_{r,s}$

All cross-edges; no edges within a part.

$$m = rs.$$

Degree s on the r-side;
degree r on the s-side.

SECTION 3

Case studies

Case study 1: Must two degrees be equal?

Problem

In a group of $n \geq 2$ people, can you find two people who have the same number of friends within the group?

Case study 1: Must two degrees be equal?

Problem

In a group of $n \geq 2$ people, can you find two people who have the same number of friends within the group?

Graph statement

In every simple graph $G = (V, E)$ with $n \geq 2$, we can find distinct $u, v \in V$ such that $\deg(u) = \deg(v)$.

$$\deg(v) \in \{0, 1, 2, \ldots, n-1\}.$$

There are n vertices and n possible values. What can we rule out?

Case study 1: Two degrees must be equal

Key restriction

Degree 0 and degree $n-1$ cannot both occur.

A vertex adjacent to everyone would also be adjacent to the isolated vertex.

Case study 1: Two degrees must be equal

Key restriction

Degree 0 and degree $n-1$ cannot both occur.

A vertex adjacent to everyone would also be adjacent to the isolated vertex.

$0, 1, \ldots, n-2$ or $1, 2, \ldots, n-1$

At most $n-1$ degree values for n vertices.

Pigeonhole principle: two vertices have the same degree.

Case study 2: The six-person theorem

Theorem

Among any six people, there are three mutual friends or three mutual strangers.

An edge joins two friends.

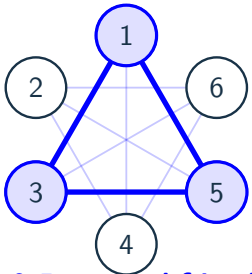


Case study 2: The six-person theorem

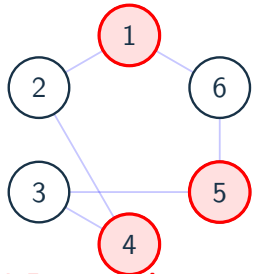
Theorem

Among any six people, there are three mutual friends or three mutual strangers.

An edge joins two friends.



$1, 3, 5$: **mutual friends**



$1, 4, 5$: **mutual strangers**

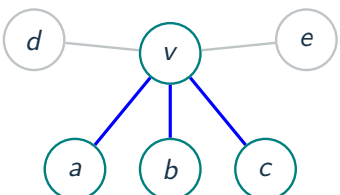
Case study 2: Focus on one person

Consider a vertex v. Each of the other five people is a friend or a stranger to v.

Pigeonhole principle

At least three are friends of v, or at least three are strangers to v.

Call them a, b, c. The same argument works with the two relations interchanged.



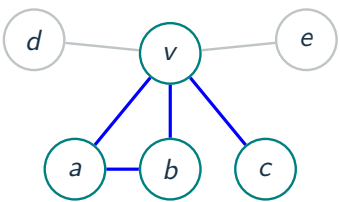
Case study 2: Focus on one person

Consider a vertex v. Each of the other five people is a friend or a stranger to v.

Pigeonhole principle

At least three are friends of v, or at least three are strangers to v.

Call them a, b, c. The same argument works with the two relations interchanged.



Some pair among a, b, c is blue

That pair and v form a blue triangle.

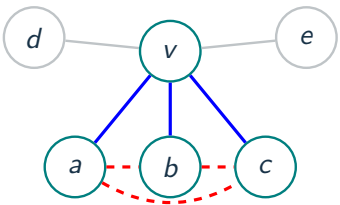
Case study 2: Focus on one person

Consider a vertex v. Each of the other five people is a friend or a stranger to v.

Pigeonhole principle

At least three are friends of v, or at least three are strangers to v.

Call them a, b, c. The same argument works with the two relations interchanged.



Some pair among a, b, c is blue

That pair and v form a blue triangle.

No pair among a, b, c is blue

All three pairs are red: a red triangle.

Case study 2: Why six people?

Think about it

Why does the argument not work for five people?

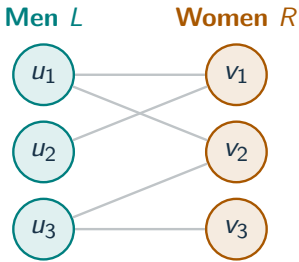
SECTION 4

Matching and Hall's theorem

Bipartite matching

MUST UNDERSTAND

Let G be a bipartite graph with parts L, R and edges between them.
An edge between a male and a female means they are mutually interested.

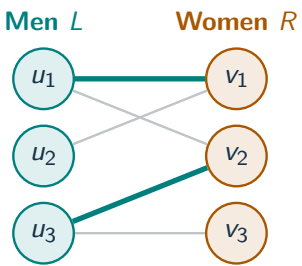


Can we pair **every participant** with **exactly one** mutually desired partner?

Bipartite matching

MUST UNDERSTAND

Let G be a bipartite graph with parts L, R and edges between them.
An edge between a male and a female means they are mutually interested.



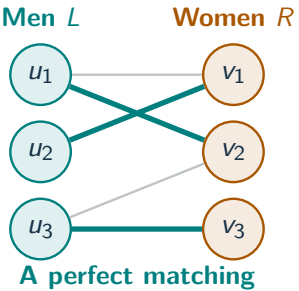
Matching

Use each vertex **at most once**.

Bipartite matching

MUST UNDERSTAND

Let G be a bipartite graph with parts L, R and edges between them.
An edge between a male and a female means they are mutually interested.



Matching

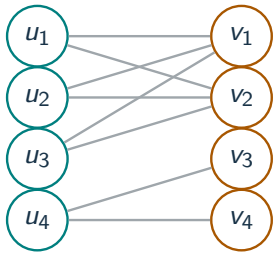
Use each vertex **at most once**.

Perfect matching

Every vertex is used **exactly once**.

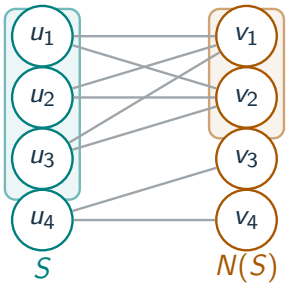
A **maximum matching** uses as many edges as possible.

A necessary condition for perfect matching



Can we match every vertex in L?

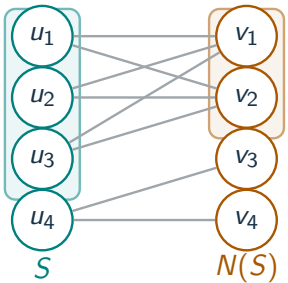
A necessary condition for perfect matching



Can we match every vertex in L?

These **3 vertices in L** have only **2 neighbors in R**.

A necessary condition for perfect matching



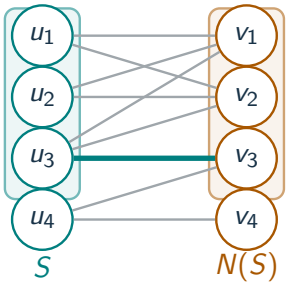
Can we match every vertex in L?

These **3 vertices in L** have only **2 neighbors in R**.

$$\underbrace{|N(S)|}_{2} < \underbrace{|S|}_{3}.$$

By the **pigeonhole principle**, pairing all three forces two to share a partner.

A necessary condition for perfect matching



Necessary condition for a perfect matching

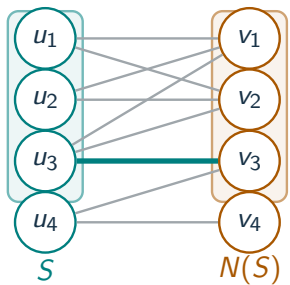
Every subset $S \subseteq L$ must have at least as many neighbors as vertices:

$$|N(S)| \geq |S|.$$

Hall's theorem

MUST UNDERSTAND

For $S \subseteq L$, $N(S)$ is its **neighbor set** (the union of its vertices' neighbors).



This condition is also sufficient

Let G be a finite bipartite graph with parts L, R.

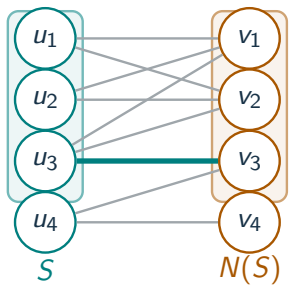
If every subset $S \subseteq L$ satisfies

$$|N(S)| \geq |S|,$$

Hall's theorem

MUST UNDERSTAND

For $S \subseteq L$, $N(S)$ is its **neighbor set** (the union of its vertices' neighbors).



This condition is also sufficient

Let G be a finite bipartite graph with parts L, R.

If every subset $S \subseteq L$ satisfies

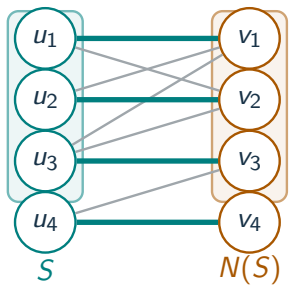
$$|N(S)| \geq |S|,$$

then G has a matching covering **every vertex in** L.

Hall's theorem

MUST UNDERSTAND

For $S \subseteq L$, $N(S)$ is its **neighbor set** (the union of its vertices' neighbors).



This condition is also sufficient

Let G be a finite bipartite graph with parts L, R.

If every subset $S \subseteq L$ satisfies

$$|N(S)| \geq |S|,$$

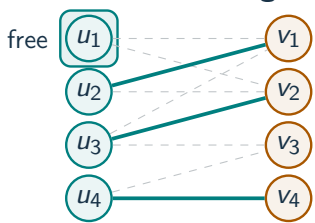
then G has a matching covering **every vertex in** L.

If $|L| = |R|$, this is a **perfect matching**.

Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.
Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.
Find an alternating path to an unmatched vertex in R, then switch its edges.

Before: 3 matching edges



Hall guarantees a new right neighbor

$S = \{u_1\}$, no right vertex reached.

$$|N(S)| \geq 1.$$

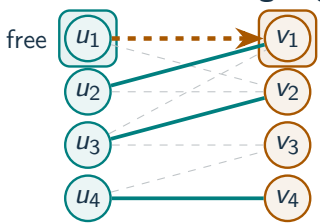
There must be a new right neighbor.

Alternating path: unused edge to R, matching edge back to L.

Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.
Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.
Find an alternating path to an unmatched vertex in R, then switch its edges.

Before: 3 matching edges



Choose the new neighbor v_1: $u_1 \to v_1$.

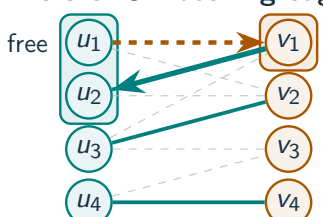
v_1 **is matched.**

Follow its matching edge back to u_2.

Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.
Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.
Find an alternating path to an unmatched vertex in R, then switch its edges.

Before: 3 matching edges



Follow the matching edge: $u_1 \to v_1 \to u_2$.

Hall guarantees a new right neighbor

$S = \{u_1, u_2\}$, only v_1 reached.

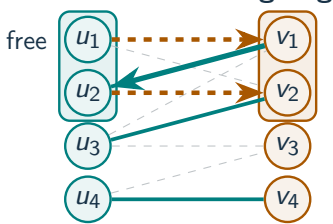
$$|N(S)| \geq 2.$$

There must be a new right neighbor.

Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.
Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.
Find an alternating path to an unmatched vertex in R, then switch its edges.

Before: 3 matching edges



v_2 **is matched.**

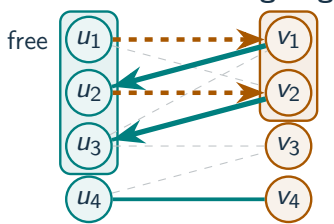
Follow its matching edge back to u_3.

Choose the new neighbor v_2: $u_1 \to v_1 \to u_2 \to v_2$.

Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.
Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.
Find an alternating path to an unmatched vertex in R, then switch its edges.

Before: 3 matching edges



Hall guarantees a new right neighbor

$S = \{u_1, u_2, u_3\}$; v_1, v_2 reached.

$$|N(S)| \geq 3.$$

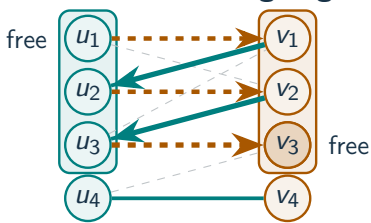
There must be a new right neighbor.

Follow the matching edge to u_3. The reached left set now has three vertices.

Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.
Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.
Find an alternating path to an unmatched vertex in R, then switch its edges.

Before: 3 matching edges



v_3 is unmatched: stop!

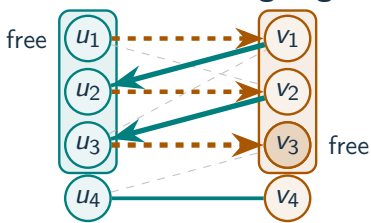
The alternating path now has two unmatched endpoints.

Here the new neighbor is v_3, reached via u_3v_3.

Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.
Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.
Find an alternating path to an unmatched vertex in R, then switch its edges.

Before: 3 matching edges



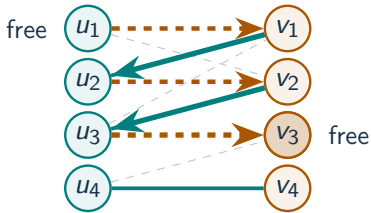
Hall's condition guarantees progress

The search can always continue until an unmatched vertex in R is found.

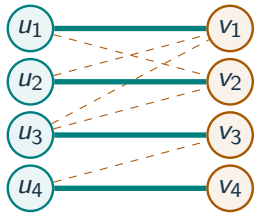
Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.
Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.
Find an alternating path to an unmatched vertex in R, then switch its edges.

Before: 3 matching edges



After: 4 matching edges



Switch the path edges: $|M| \longrightarrow |M| + 1$, contradicting that M is maximum.

Proof of Hall's theorem

Assume $|N(S)| \geq |S|$ **for every** $S \subseteq L$.

Let M be a **maximum matching**, with an unmatched vertex $u_1 \in L$.

Fin

M must match every vertex in L.

When $|L| = |R|$, we have proved that

M is a perfect matching!

Switch the path edges: $|M| \longrightarrow |M| + 1$, contradicting that M is maximum.

Practice: Hall's theorem

A finite simple bipartite graph has parts L, R. Every vertex has degree $d \geq 1$.

Prove

There is a matching covering every vertex in L.

Practice: Hall's theorem

A finite simple bipartite graph has parts L, R. Every vertex has degree $d \geq 1$.

Prove

There is a matching covering every vertex in L.

Hint: For an arbitrary $S \subseteq L$, count the edges from S to $N(S)$.

Summary

Graph models	Vertices represent objects; edges represent relationships.
Degree arguments	Use degree sums and the pigeonhole principle to prove graph properties.
Matching	Choose edges with no shared endpoints. Hall's condition guarantees a matching covering every vertex in one part.

SECTION 5

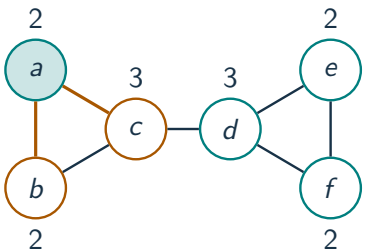
Optional case studies

Optional case study 3: Friendship paradox

Do you feel like your friends are more popular than you?

Optional case study 3: Friendship paradox

Do you feel like your friends are more popular than you?



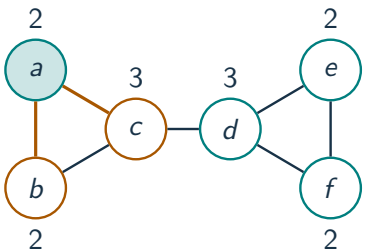
Numbers beside vertices are degrees.

You are *a*

Your number of friends: 2.
Your friends: *b* and *c*.

Optional case study 3: Friendship paradox

Do you feel like your friends are more popular than you?



Numbers beside vertices are degrees.

You are a

Your number of friends: 2.
Your friends: b and c.

Your friends' average

$$\frac{\deg(b) + \deg(c)}{2} = \frac{2+3}{2} = 2.5 > 2.$$

Case study 3: A surprising theorem (friendship paradox)

For any simple undirected graph with no isolated vertices.

Theorem

Choose a person uniformly at random. In expectation, their friends' average number of friends is at least their own number of friends.

$$\mathbb{E}_v[f(v)] \geq \mathbb{E}_v[\deg(v)].$$

Here $f(v) = \dfrac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$ is your friends' average.

Case study 3: A surprising theorem (friendship paradox)

For any simple undirected graph with no isolated vertices.

Theorem

Choose a person uniformly at random. In expectation, their friends' average number of friends is at least their own number of friends.

$$\mathbb{E}_v[f(v)] \geq \mathbb{E}_v[\deg(v)].$$

Here $f(v) = \dfrac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$ is your friends' average.

Intuition: A popular person appears in more people's friend lists, so their degree enters more people's averages.

Case study 3: From expectations to sums

Choose v uniformly from V.

Goal: $\mathbb{E}_v[f(v)] \geq \mathbb{E}_v[\deg(v)]$.

$$\mathbb{E}_v[f(v)] = \sum_{v \in V} \underbrace{\frac{1}{n}}_{\text{prob.}} \cdot \underbrace{f(v)}_{\text{value}} = \frac{1}{n} \sum_{v \in V} f(v),$$

$$\mathbb{E}_v[\deg(v)] = \sum_{v \in V} \frac{1}{n} \cdot \deg(v) = \frac{1}{n} \sum_{v \in V} \deg(v).$$

Case study 3: From expectations to sums

Choose v uniformly from V.

Goal: $\mathbb{E}_v[f(v)] \geq \mathbb{E}_v[\deg(v)]$.

$$\mathbb{E}_v[f(v)] = \sum_{v \in V} \underbrace{\frac{1}{n}}_{\text{prob.}} \cdot \underbrace{f(v)}_{\text{value}} = \frac{1}{n} \sum_{v \in V} f(v),$$

$$\mathbb{E}_v[\deg(v)] = \sum_{v \in V} \frac{1}{n} \cdot \deg(v) = \frac{1}{n} \sum_{v \in V} \deg(v).$$

It suffices to prove:

$$\sum_{v \in V} f(v) \geq \underbrace{\sum_{v \in V} \deg(v) = 2m}_{\text{handshaking theorem}}.$$

Case study 3: The formal proof

1. Enumerate vertices, then their neighbors.

$$f(v) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$$

The average degree of v's friends.

$$\sum_{v \in V} f(v) = \sum_{v \in V} \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$$

Case study 3: The formal proof

1. Enumerate vertices, then their neighbors.

$$f(v) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$$

The average degree of v's friends.

$$\begin{aligned} \sum_{v \in V} f(v) &= \sum_{v \in V} \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u) \\ &= \sum_{v \in V} \sum_{u \in N(v)} \frac{\deg(u)}{\deg(v)}. \end{aligned}$$

Case study 3: The formal proof

1. Enumerate vertices, then their neighbors.

$$f(v) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$$

The average degree of v's friends.

$$\begin{aligned} \sum_{v \in V} f(v) &= \sum_{v \in V} \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u) \\ &= \sum_{v \in V} \sum_{u \in N(v)} \frac{\deg(u)}{\deg(v)}. \end{aligned}$$

2. Enumerate edges instead.

Each edge $\{u, v\}$ appears twice: from v to u, and from u to v.

$$\sum_{v \in V} f(v) = \sum_{\{u,v\} \in E} \left(\frac{\deg(u)}{\deg(v)} + \frac{\deg(v)}{\deg(u)} \right).$$

Case study 3: The formal proof (continued)

$$f(v) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$$

The average degree of v's friends.

Recall: for positive integers a, b, $\frac{a}{b} + \frac{b}{a} \geq 2$.

$$\sum_{v \in V} f(v) = \sum_{\{u,v\} \in E} \left(\frac{\deg(u)}{\deg(v)} + \frac{\deg(v)}{\deg(u)} \right)$$

Case study 3: The formal proof (continued)

$$f(v) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$$

The average degree of v's friends.

Recall: for positive integers a, b, $\frac{a}{b} + \frac{b}{a} \geq 2$.

$$\sum_{v \in V} f(v) = \sum_{\{u,v\} \in E} \left(\frac{\deg(u)}{\deg(v)} + \frac{\deg(v)}{\deg(u)} \right) \geq \sum_{\{u,v\} \in E} 2 = 2m.$$

Case study 3: The formal proof (continued)

$$f(v) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$$

The average degree of v's friends.

Recall: for positive integers a, b, $\frac{a}{b} + \frac{b}{a} \geq 2$.

$$\sum_{v \in V} f(v) = \sum_{\{u,v\} \in E} \left(\frac{\deg(u)}{\deg(v)} + \frac{\deg(v)}{\deg(u)} \right) \geq \sum_{\{u,v\} \in E} 2 = 2m.$$

This proves the required sum inequality. Dividing by n gives

$$\mathbb{E}_v[f(v)] \geq \mathbb{E}_v[\deg(v)].$$

Case study 3: The formal proof (continued)

$$f(v) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \deg(u)$$

The average degree of v's friends.

Recall: for positive integers a, b, $\frac{a}{b} + \frac{b}{a} \geq 2$.

$$\sum_{v \in V} f(v) = \sum_{\{u,v\} \in E} \left(\frac{\deg(u)}{\deg(v)} + \frac{\deg(v)}{\deg(u)} \right) \geq \sum_{\{u,v\} \in E} 2 = 2m.$$

This proves the required sum inequality. Dividing by n gives

$$\mathbb{E}_v[f(v)] \geq \mathbb{E}_v[\deg(v)].$$

Think: Why did we exclude isolated vertices?

Optional case study 4: Triangle-free graphs

Problem

What is the maximum number of edges in a simple undirected graph on $n \geq 1$ vertices with no triangle?

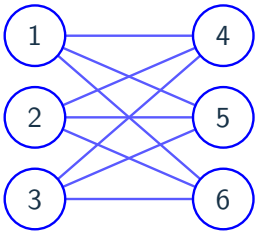
Triangle-free: no cycle of length three.

Optional case study 4: Triangle-free graphs

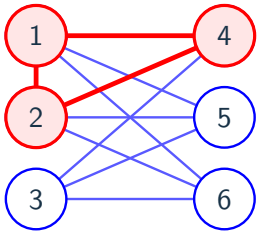
Problem

What is the maximum number of edges in a simple undirected graph on $n \geq 1$ vertices with no triangle?

Triangle-free: no cycle of length three.



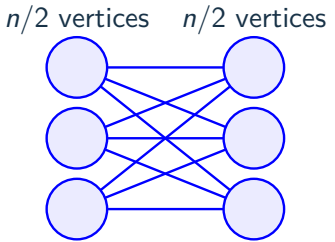
Valid: no triangle



Invalid: contains a triangle

Case study 4: How well can we do?

Bipartite graphs have no triangles.



Balanced complete bipartite graph

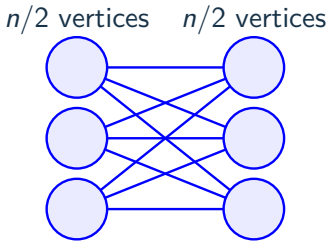
(even n)

$$m = \frac{n}{2} \times \frac{n}{2} = \frac{n^2}{4}.$$

Best among bipartite graphs.

Case study 4: How well can we do?

Bipartite graphs have no triangles.



Balanced complete bipartite graph

(even n)

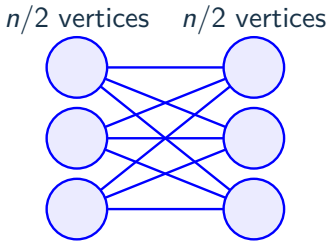
$$m = \frac{n}{2} \times \frac{n}{2} = \frac{n^2}{4}.$$

Best among bipartite graphs.

Can a triangle-free graph do better?

Case study 4: How well can we do?

Bipartite graphs have no triangles.



Balanced complete bipartite graph

(even n)

$$m = \frac{n}{2} \times \frac{n}{2} = \frac{n^2}{4}.$$

Best among bipartite graphs.

Can a triangle-free graph do better?

Mantel's theorem

A triangle-free simple graph on n vertices satisfies $m \leq \lfloor n^2/4 \rfloor$.

Case study 4: Prove that $n^2/4$ is an upper bound

Let's consider what "no triangle" means.

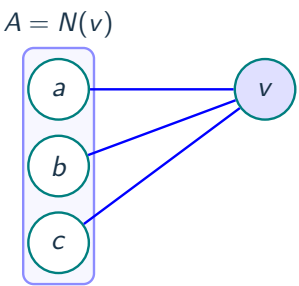
Pick **any vertex** v.



Case study 4: Prove that $n^2/4$ is an upper bound

Let's consider what "no triangle" means.

Pick **any vertex** v. Put $A = N(v)$ on the left

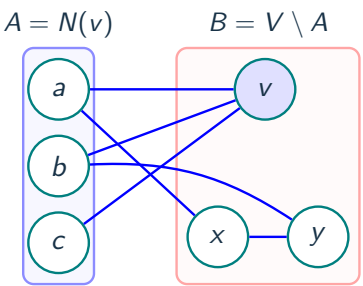


Every vertex in A is adjacent to v.

Case study 4: Prove that $n^2/4$ is an upper bound

Let's consider what "no triangle" means.

Pick **any vertex** v. Put $A = N(v)$ on the left and $B = V \setminus A$ on the right.

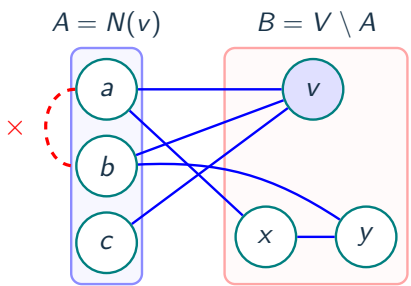


Every vertex in A is adjacent to v.

Case study 4: Prove that $n^2/4$ is an upper bound

Let's consider what "no triangle" means.

Pick **any vertex** v. Put $A = N(v)$ on the left and $B = V \setminus A$ on the right.



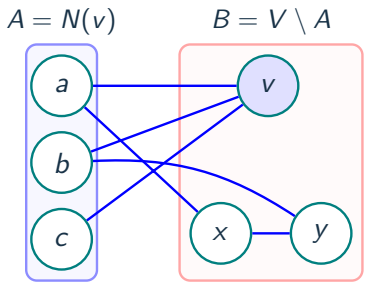
Every vertex in A is adjacent to v.

An edge inside A would form a triangle with v.

There are no edges inside A.

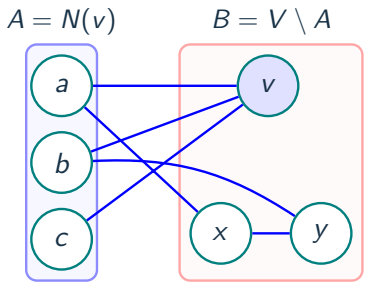
So every edge touches at least one vertex in B.

Case study 4: Count edges through the other part



So every edge touches at least one vertex in B.

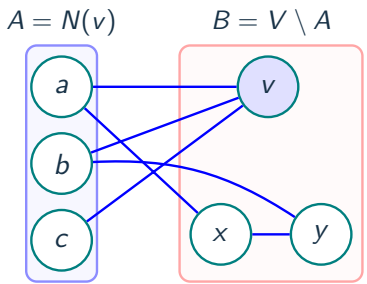
Case study 4: Count edges through the other part



So every edge touches at least one vertex in B.

$$m \leq \sum_{u \in B} \deg(u).$$

Case study 4: Count edges through the other part



So every edge touches at least one vertex in B.

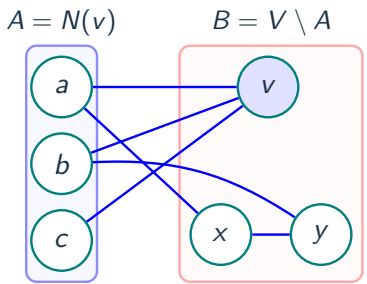
$$m \leq \sum_{u \in B} \deg(u).$$

Let Δ be the maximum degree in the graph.

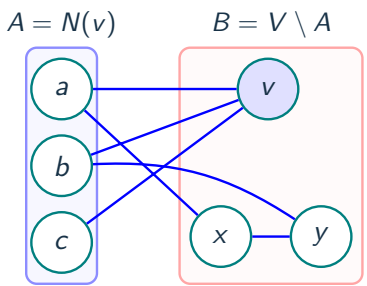
$$\begin{aligned} \sum_{u \in B} \deg(u) &\leq \sum_{u \in B} \Delta \\ &= (n - \deg(v))\Delta. \end{aligned}$$

Case study 4: Choose a maximum-degree vertex

$$m \leq (n - \deg(v))\Delta.$$



Case study 4: Choose a maximum-degree vertex

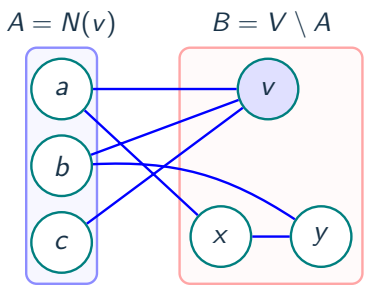


$$\boldsymbol{m \leq (n - \deg(v))\Delta.}$$

Now choose v with maximum degree:

$$\deg(v) = \Delta.$$

Case study 4: Choose a maximum-degree vertex



$$m \leq (n - \deg(v))\Delta.$$

Now choose v with maximum degree:

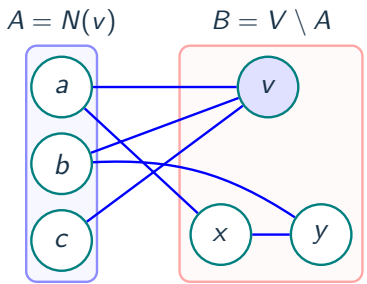
$$\deg(v) = \Delta.$$

An upper bound on the number of edges

$$m \leq (n - \Delta)\Delta.$$

Case study 4: Finish the upper bound

$$m \leq (n - \Delta)\Delta.$$



Lemma (prove it yourself)

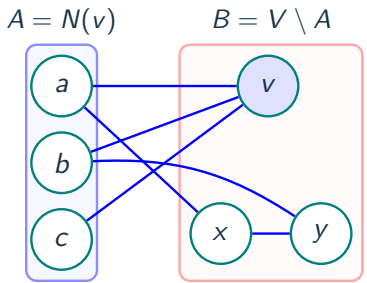
If $a + b = n$, then

$$ab \leq \frac{n^2}{4}.$$

Equality holds when $a = b = n/2$.

Case study 4: Finish the upper bound

$$m \leq (n - \Delta)\Delta.$$



Lemma (prove it yourself)

If $a + b = n$, then

$$ab \leq \frac{n^2}{4}.$$

Equality holds when $a = b = n/2$.

Take $a = n - \Delta$ and $b = \Delta$.

$$\boldsymbol{m \leq (n - \Delta)\Delta \leq \frac{n^2}{4}.}$$